S.SINHA COLLEGE AURANGABAD (BIHAR)

P.G NOTES

TOPIC NAME :- REGULAR & NORMAL SPACES

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Regular space:-

A topological space X is called a regular space if for every closed subset F of X and every point $x \in X$ s.t. $x \notin F$, $\exists$ disjoint open sets G and H s.t. $F \subseteq G$ and $x \in H$.

Ex:- Let $X = \{x, y, z\}$, we consider the topology T on X given by $T = \{\phi, X, \{x\}, \{y, z\}\}$. Since the closed subsets of the topological space (X, T) are $X, \phi, \{y, z\}, \{x\}$. So it is a regular space. However (X, T) is not a T_1-space, since the singleton $\{z\}$ is not closed.

Thus a regular space which is also a T_1-space is called a T_3-space.

Q. Show that every T_3-space is a T_2-space.

Ans:- Let X be a T_3-space. Let $x, y \in X$ and $x \neq y$. Since X is a T_1-space, $\{x\}$ is closed and since $y \neq x$, $y \notin \{x\}$.

Since X is regular, therefore $\exists$ disjoint open sets G and H s.t.

$$\{x\} \subseteq G \text{ and } y \in H.$$

Thus $x \in G$ and $y \in H$.

Thus x and y belongs respectively to the disjoint open sets G and H. Hence X is T_2-space.

Normal space:-

A topological space X is called a normal space if for ~~such~~ each pair of disjoint closed subsets F_1 and F_2 of X, $\exists$ a pair of disjoint open sets G and H s.t.

$$F_1 \subseteq G \text{ and } F_2 \subseteq H.$$

Theorem:- Show that every metric space is normal.

Ans- Let (X,d) be a metric space. Let A and B be disjoint closed subsets of X, we shall show that $\exists$ disjoint open sets G and H s.t. $A \subseteq G$ and $B \subseteq H$.

If either A or B is empty, say $A = \phi$ then ϕ and X are disjoint open sets s.t. $A \subseteq \phi$ and $B \subseteq X$.

Hence we may assume that A and B are non empty. Let $a \in A$. Since A and B are disjoint $a \notin B$.

Since B is closed $d(a,B) = \delta_a > 0$

similarly, if $b \in B$ then $d(b,A) = \delta_b > 0$.

let $S_a = S(a, \frac{\delta_a}{4})$, $S_b = S(b, \frac{\delta_b}{4})$ so $a \in S_a$, $b \in S_b$.

We take $G = \bigcup_{a \in A} S_a$, $H = \bigcup_{b \in B} S_b$;

Now, G and H are open sets, since each of G and H is a union of open spheres.

Since $a \in S_a$, $b \in S_b$ $\forall a \in A, b \in B$, it follows that $A \subseteq G$ and $B \subseteq H$.

We next show that $G \cap H = \phi$. Suppose if possible $G \cap H \neq \phi$.

Let $z \in G \cap H$. Then $z \in G$, $z \in H$. Hence $\exists$ $a_0 \in A$, $b_0 \in B$ s.t. $z \in S_{a_0}$, $z \in S_{b_0}$. Let $d(a_0, b_0) = \alpha > 0$.

then $d(a_0, B) = \delta_{a_0} \leq \alpha$ and $d(b_0, A) = \delta_{b_0} \leq \alpha$.

But $z \in S_{a_0}$ and $z \in S_{b_0}$ so $d(a_0, z) < \frac{1}{4}\delta_{a_0}$ and $d(z, b_0) < \frac{1}{4}\delta_{b_0}$.

Hence by the triangle inequality

$d(a_0, b_0) = \alpha \leq d(a_0, z) + d(z, b_0) = \frac{1}{2}\alpha$.

Thus $2\alpha < \alpha$ and $2 < 1$, which is impossible. Hence $G \cap H = \phi$

Q. Show that every compact Hausdorff space is a normal T_1-space. i.e a T_4-space.

Ans:- Let X be a compact Hausdorff space. Since X is a T_2-space, X is also a T_1-space. Now, in order to show that X is a normal space, let A and B be any two disjoint closed subsets of X.

If atleast one of A, B is ϕ, say $A = \phi$ we can take $G = \phi$, $H = X$ then G and H are disjoint open sets such that $A \subseteq G$, $B \subseteq H$. We may, therefore assume that both A and B are non-empty.

Since X is compact, A and B are disjoint compact subspaces of X. Let p be a point of A, then $p \notin B$. Since X is Hausdorff, then $\exists$ disjoint open sets G_p, H s.t. $p \in G_p$, $B \subseteq H_B$. If we allow p to vary over A, we obtain a class of G_p whose union contains A; and since A is compact, some finite subclass which we denote by

$\{G_1, G_2, \ldots, G_n\}$ is s.t. $A \subseteq \bigcup_{i=1}^{n} G_i$. If $H_1, H_2, \ldots, H_n$ are the H_B's which correspond to $G_1, G_2, \ldots, G_n$, we put

$$G = \bigcup_{i=1}^{n} G_i, \quad H = \bigcap_{i=1}^{n} H_i.$$

then G and H are open sets s.t. $A \subseteq G$, $B \subseteq H$.

$\therefore$ $G \cap H = \phi$.

$\therefore$ X is a normal space. Thus X is a normal T_1-space. i.e T_4-space.

Theorem:- A topological space X is normal iff for every closed set F and open set H containing F $\exists$ an open set G s.t.

$$F \subseteq G \subseteq \bar{G} \subseteq H.$$

Proof:- Necessary condition:- Let X be a normal space, let F be a closed set in X and H an open set such that $F \subseteq H$. Then H^c is closed and $F \cap H^c = \phi$.

Thus F and H^c are disjoint closed sets in the normal space X. Hence $\exists$ open sets G and M such that $F \subseteq G$, $H^c \subseteq M$ and $G \cap M = \phi$.

Now, $G \cap M = \phi \Rightarrow G \subseteq M^c$ and $H^c \subseteq M = M \subseteq H$.

Moreover, M^c is closed.

Hence $F \subseteq G \subseteq \bar{G} \subseteq M \subseteq H$.

Sufficient condition:-

Suppose that the given condition is satisfied. Let F_1 and F_2 be any two disjoint closed sets in X. Then $F_1 \subseteq F_1^c$ and F_2^c is open.

Hence by the given condition $\exists$ an open set G s.t.

$$F_1 \subseteq G \subseteq \bar{G} \subseteq F_2^c.$$

But $\bar{G} \subseteq \bar{F_2} \Rightarrow F_2 \subseteq \bar{G}^c$ and $G \subseteq \bar{G} \Rightarrow G \cap \bar{G}^c = \phi$.

Moreover, $\bar{G}^c$ is open.

Thus, $F_1 \subseteq G$, $F_2 \subseteq \bar{G}^c$ with $G, \bar{G}^c$ disjoint open sets,

Hence X is normal.

Theorem:- show that a closed subspace of normal space is normal.

Ans:- Let M be a closed subspace of a normal space X, Let A and B be any two disjoint closed subsets of M, then A and B are also disjoint closed subsets of X.
Since X is normal therefore exist disjoint open sets G and H in X s.t. $A \subseteq G$ and $B \subseteq H$,
then $M \cap G$ and $M \cap H$ are disjoint open sets in M s.t. $A \subseteq M \cap G$, $B \subseteq M \cap H$.
Hence M is normal.

*. Urysohn's lemma :-

Q. state and prove Urysohn's lemma.

Ans:- Statement:- If A and B are disjoint closed subsets of a normal space X then $\exists$ a continuous real function f defined on X with $0 \le f(x) \le 1$ for all $x \in X$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof:- Since $A \cap B = \phi$, $A \subseteq B^c$, Since B is closed, then B^c is an open set containing the closed set A, since X is normal space, it follows that $\exists$ an open set which we name $G_{\frac{1}{2}}$ s.t.

$$A \subseteq G_{\frac{1}{2}} \subseteq \overline{G}_{\frac{1}{2}} \subseteq B^c.$$

Now, $G_{\frac{1}{2}}$ is an open set containing the closed set A and B^c is an open set containing the closed set $\overline{G}_{\frac{1}{2}}$.

P.T.O.

Urysohn's lemma (continue):-

We now prove that $f: X \to [0,1]$ is continuous. For this we note that all intervals of the form $[0, a[$ and $]b, 1]$ (where $0<a, b<1$) constitute an open subbase for the subspace $[0,1]$ of the real line R.

In order to show that f is continuous, it is sufficient to prove that $f^{-1}([0,a[)$ and $f^{-1}(]b,1])$ are ~~an~~ open sets in X (where $0<a, b<1$).

We first show that,

$$f(x) < a \Leftrightarrow x \in G_l \text{ for some } l<a.$$

In fact, if $x \notin G_l$ for any $l<a$, then since S is dense in $[0,1]$, it follows from the definition of f that $f(x) = \text{l.u.b}\{l : x \notin G_l\} \geq a$.

Hence $f(x) < a \Rightarrow$ that $x \in G_l$ for some $l<a$.

On the other hand, if $f(x) \geq a$ and $l<a$ then again by defn. of l.u.b. $\exists$ l_1 in S s.t. $l < l_1 \leq a$. and s.t. $x \notin G_{l_1}$.

Thus if $f(x) \geq a$, $x \notin G_l$, for every $l<a$.

Hence if $x \in G_l$ for some $l<a$ then $f(x)<a$.

Thus the element holds. It follows that

$$\begin{aligned} f^{-1}([0,a[) &= \{x \in X : 0 \leq f(x) < a\}. \\ &= \{x \in X : x \in G_l \text{ for some } l<a\}. \\ &= \bigcup_{l<a} G_l \end{aligned}$$

Since each G_l is an open set and so any union of open sets is open.

It follows that $\bigcup_{l<a} G_l$ is open in X. Hence $f^{-1}([0,a[)$ is an open set in X.

P.T.O.

Urysohn's lemma (Remaining part):-

Since X is a normal space, $\exists$ open sets which we name as $G_{\frac{1}{2}}, G_{\frac{3}{4}}$ s.t.

$$A \subseteq G_{\frac{1}{4}} \subseteq \bar{G}_{\frac{1}{4}} \subseteq G_{\frac{1}{2}} \subseteq \bar{G}_{\frac{1}{2}} \subseteq G_{\frac{3}{4}} \subseteq \bar{G}_{\frac{3}{4}} \subseteq B^c.$$

continuing this process, for each dyadic numbers of the form $l = \frac{m}{2^n}$ where $n = 1, 2, 3, \ldots$, and $m = 1, 2, 3, \ldots, 2^{n-1}$ of $[0,1]$, we can name an open set G_i with the property that for any two dyadic rational numbers l_1 and l_2 in $[0,1]$.

$$l_1 < l_2 \Rightarrow A \subseteq G_{l_1} \subseteq \bar{G} \subseteq G_{l_2} \subseteq \bar{G}_{l_2} \subseteq B^c.$$

It may be noted that the set S of all dyadic rational numbers in $[0,1]$ is dense in $[0,1]$.

We now define a function f on X by

$$f(x) = 0 \text{ if } x \in \text{every } G_i$$
$$= \text{l.u.b.} \{l : x \notin G_i\} \text{ otherwise.}$$

Since $0 \leq l \leq 1$, it is clear that $0 \leq f(x) \leq 1$.

Also, if $x \in A$ then by the compositions of G_i, $x \in$ every G_i and hence $f(x) = 0$.

again, if $y \in B$, then since every $G_i \subseteq B^c$, $y \notin$ any G_i.

Hence $f(y) = \text{l.u.b.} \{l : y \notin G\}$.

$= $ l.u.b of all the dyadic rational numbers l in $[0,1]$.

$= \text{l.u.b } S$

$= 1$ (since S is dense in $[0,1]$).

Thus $f(x) = 0$ for $x \in A$ and $f(y) = 1$ for $y \in B$.

Hence $f(A) = \{f(x) : x \in A\} = \{0\}$ and

$f(B) = \{f(y) : y \in B\} = \{1\}$.

P.T.O.

Remaining part of Urysohn's lemma:-

It can also be seen that

$$f(x) > b \Leftrightarrow x \in \overline{G}_l \text{ for some } l > b.$$

Hence $f^{-1}([b,1]) = \{x \in X : f(x) > b\}$.

$$= \bigcup_{l > b} \overline{G}_l^{\,c}.$$

since each $\overline{G}_l^{\,c}$ is open, it follows that $f^{-1}([b,1])$ is also an open set in X. Hence f is continuous.

~~This~~ Hence the proof.

The Tietze Extension Theorem

Q. State and prove Tietze Extension theorem.

Ans:- Statement:- Any bounded continuous real function on a closed subset F of a normal space X can be extended continuously to the whole space X; preserving the same bounds.

Proof:- Let f be continuous on the closed subset F of X and let $f(x) \leq K \ \forall x \in F$ and for some real number $K > 0$.

Consider the subsets F_1 and F_2 of F defined by $F_1 = \{x \in F : f(x) \leq \frac{-K}{3}\} = f^{-1}[-K, \frac{-K}{3}]$

and $F_2 = \{x \in F : \overline{f}(x) \geq \frac{K}{3}\} = f^{-1}[\frac{K}{3}, K]$.

Then F_1 and F_2 are disjoint, non-empty and closed in F, and since F is closed in X, F_1 and F_2 are also closed in X. Since X is normal, it follows that $\exists$ a continuous function

$e_1' : X \to [\frac{-K}{3}, \frac{K}{3}]$ s.t.

$e_1(F_1) = \{\frac{-K}{3}\}$ and $e_1(F_2) = \{\frac{K}{3}\}$.

P.T.O.

Tietze Extension Theorem (Remaining part):-

We next define a function h_1 on F by $h_1(x) = f(x) - g_1(x)$. Since f and g_1 are continuous, h_1 is also continuous. Moreover, $|h_1(x)| \leq \frac{2}{3}K$.

This can be seen as follows:-

(i) If $x \in F_1$ then $-K \leq f(x) \leq -\frac{K}{3}$ and $g_1(x) = -\frac{K}{3}$.
Hence $-\frac{2}{3}K \leq f(x) - g_1(x) \leq 0$.

i.e $-\frac{2}{3}K \leq h_1(x) \leq 0 \leq \frac{2}{3}K$.

(ii) If $x \in F_2$ then $\frac{K}{3} \leq f(x) \leq K$ and
$g_1(x) = \frac{K}{3}$.

Hence $\frac{K}{3} - \frac{K}{3} \leq f(x) - g_1(x) \leq K - \frac{K}{3}$.

i.e $0 \leq h_1(x) \leq \frac{2}{3}K$.

(iii) Lastly, if $x \in F$ but $x \notin F_1 \cup F_2$.

then $-\frac{K}{3} < f(x) < \frac{K}{3}$ and $-\frac{K}{3} \leq g_1(x) \leq \frac{K}{3}$,

so that $-\frac{K}{3} - \frac{K}{3} < f(x) - g_1(x) < \frac{K}{3} - \left(-\frac{K}{3}\right)$.

Thus $-\frac{2K}{3} < f(x) - g_1(x) = h_1(x) < \frac{2K}{3}$.

Thus in every case, $-\frac{2}{3}K \leq h_1(x) \leq \frac{2}{3}K$.

i.e $|h_1(x)| \leq \frac{2}{3}K$ for all $x \in F$.

Now, applying the above procedure to $h_1(x)$ with bounds $-\frac{2K}{3}$ and $\frac{2}{3}K$, a continuous functions $g_2(x)$ is obtained on the whole space X with $|g_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3}K$ and a continuous fn $h_2(x) = h_1(x) - g_2(x)$ is defined on F with $|h_2(x)| \leq \left(\frac{2}{3}\right)^2 K$. P.T.O.

Tietze Extension Theorem:(Remaining part):-

In general, we obtain for each +ve integer n, a continuous fn $g_n(x)$ on X with $|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} K$, and a continuous function $h_n(x) = h_{n-1}(x) - g_n(x)$ on F with $|h_n(x)| \le \left(\frac{2}{3}\right)^n K$.

Now, by Weierstrass's M-test the infinite series $\sum_{n=1}^{\infty} g_n(x)$ of continuous fn converges uniformly on X, and so defines a continuous function $f_0(x)$ on X with

$$|f_0(x)| \le \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n K = K.$$

Also, on F we have

$$\begin{aligned} f_0(x) &= g_1(x) + \sum_{n=1}^{\infty} g_{n+1}(x) \\ &= f(x) - h_1(x) + \sum_{n=1}^{\infty} \left(h_n(x) - h_{n+1}(x)\right) \\ &= \lim_{n\to\infty} \left(f(x) - h_{n+1}(x)\right) \end{aligned}$$

Since $|h_{n+1}(x)| \le \left(\frac{2}{3}\right)^{n+1} \cdot K$, $\lim_{n\to\infty} h_{n+1}(x) = 0$.

Hence $f_0(x) = f(x)$ on F.

Thus $\exists$ a continuous fn f_0 on X which is an extension of the given continuous bounded fn f on F and f_0 has the same bounds $-K$ and K which were the bounds for f.

Hence the theorem.

* Countably compact space:-

A topological space X is said to be countably compact if every infinite subset E of X has at least one accumulation point in X.

⇒ Eg:- Every bounded closed interval $[a,b]$ in (subspace of $\mathbb{R}$) is countably compact.

→ If E is an infinite subset of $[a,b]$ then E ~~has at least one a~~ is also bounded and hence by Bolzano-Weierstrass theorem, E has at least one accumulation point $m \in \mathbb{R}$. Since $E \subseteq [a,b]$, m is also an accumulation point of $[a,b]$. But $[a,b]$ is closed. Hence $m \in [a,b]$. Hence $[a,b]$ is countably compact.

⇒ Eg:- Every compact space is countably compact.

→ Let X be a compact space. Suppose, if possible, X is not countably compact. Then ∃ an infinite subset E of X which has no accumulation point. Hence E is closed. Take any $x \in E$. Then x is not an accumulation point of E. Hence ∃ an open set V_x containing x which contains no point of E other than x. Consider the family

$$\{E^c\} \cup \{V_x : x \in E\}$$

This family is an open covering of X which is not reducible to any finite subcovering of X. Hence X is not compact. But this contradicts our supposition that X is compact. Hence every compact space is countably compact.

* Example of a countably compact space which is not compact.

→ Let N be the set of all +ve integers and let T be the topology on N generated by the family $H = \{\{2n-1, 2n\} : n \in N\}$ of subsets of N.

Let E be a non-empty subset of N. Let $m_0 \in E$. If m_0 is even $m_0 - 1$ is an accumulation point of E and if m_0 is odd then $m_0 + 1$ is an accumulation point of E. Hence every non-empty subset of N has an accumulation point. Hence (N, T) is countably compact.

The family H is an open covering of N which is not reducible to any finite subcovering of N. Hence (N, T) is not compact.

Sequentially compact space:-

A topological space X is said to be sequentially compact if every sequence in X contain a subsequence which converges to a point of X.

Eg:- Every finite subspace of a topological space is sequentially compact.

** Every sequentially compact space is countably compact but not conversely.

→

Q. Show that every locally compact Hausdorff space is a Tychonoff space.

Soln:- Let (X, T) be a locally compact Hausdorff space and let (X_p, T_p) be the one point compactification of (X, T). We know that (X_p, T_p) is a compact Hausdorff space and that the given topology T on X equals its relative topology as a subspace of (X_p, T_p).

Since X is a T_2 space, X is also a T_1 space.

Now, let $x \in X$ and let F be any closed subset of X s.t. $x \notin F$.

By the defn of relative topology, $\exists$ a closed set H in X_p s.t. $F = X \cap H$. Then $x \notin H$.

Now, $\{x\}$ and H are disjoint closed sets in the normal space X_p. Hence by Urysohn's lemma, $\exists$ a continuous fn $f: X_p \to [0,1]$ s.t.

$$f(\{x\}) = \{0\} \text{ and } f(H) = \{1\}.$$

i.e $f(x) = 0$ and $f(H) = \{1\}$.

Since $F \subset H$, we have $f(F) = \{1\}$.

Now, the restriction f' of f to X is a continuous fn $f': X \to [0,1]$ s.t. $f'(x) = 0$ and $f'(F) = \{1\}$.

Hence (X, T) is a completely regular T_1 space.

i.e (X, T) is a Tychonoff space.

Proved.

Thank you….